Problem 1. All terms of an infinite convergent geometric sequence are nonzero. Show that the ratio of any chosen term of this sequence to the sum of all terms that follow it is constant. Find this constant ratio.

Source: Mathematics – problem collection, grade 3, extended level
Problem selection: Maja Chlewicka
Solution: Since $a_1 \cdot q \neq 0$, we have $a_1 \neq 0$ and $q \neq 0$. Moreover, $|q| < 1$ and

$$S = \frac{a_1}{1-q}.$$

Let us consider, for some $k > 0$, the quantity

$$\frac{a_k}{a_{k+1} + a_{k+2} + a_{k+3} + \dots}.$$

$$a_{k+1} + a_{k+2} + a_{k+3} + \dots = \frac{a_{k+1}}{1-q} = a_1 \cdot \frac{q^k}{1-q}.$$

$$\cancel{a_1} \cdot \frac{q^{k-1}}{\cancel{a_1}} \cdot \frac{1-q}{q^k} = q^{-1}(1-q) = \frac{1-q}{q} = \text{const}.$$

This completes the proof.

Problem 2. Nonzero real numbers a, b satisfy the condition

$$(a+b^2)(a^3+b^3) = a^4 + b^5.$$

Show that $b < 0$.

Source: 21st Junior Mathematical Olympiad, First Round – problem section
Problem selection: Maja Chlewicka
Solution: By transforming the equation we obtain:

$$(a+b^2)(a^3+b^3) = a^4 + b^5,$$

$$a(a^3+b^3) + b^2(a^3+b^3) = a^4 + b^5,$$

$$a^4 + ab^3 + a^3b^2 + b^5 = a^4 + b^5,$$

$$ab^3 + a^3b^2 = 0,$$

$$ab^2(b+a^2) = 0.$$

Since $a \neq 0$ and $b \neq 0$, the last equality implies

$$b + a^2 = 0,$$

that is $b = -a^2$. Because $a^2 > 0$ for every $a \neq 0$, we conclude that $b < 0$.