Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$2f(x) + f(1-x) = 3x$$

for every $x \in \mathbb{R}$.

Problem author: Henryk Pawłowski
Problem selection: Maria Janyska
Solution: First, substitute $x := 1 - x$.

$$2f(1-x) + f(x) = 3 - 3x$$

Now add the two equations side by side.

$$3f(x) + 3f(1-x) = 3$$

$$f(x) + f(1-x) = 1$$

Subtracting this equation from the original condition, we obtain

$$f(x) = 3x - 1$$

as the only solution of the functional equation. Let us verify whether the obtained formula satisfies the original equation.

$$2f(x) + f(1-x) = 3x$$

$$6x - 2 + 3 - 3x - 1 = 3x$$

$$3x = 3x$$

Since the equation is satisfied, the unique solution is the function $f(x) = 3x - 1$.

Problem 2. Let a, b, c be the side lengths of a triangle, p be its semiperimeter, and r be the radius of the inscribed circle. Prove that

$$\frac{1}{(p-a)^2} + \frac{1}{(p-b)^2} + \frac{1}{(p-c)^2} \geqslant \frac{1}{r^2}.$$

Problem author: Henryk Pawłowski
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Solution:
Method 1: In this solution, we use two nonstandard formulas for the area of a triangle (denoted by S).

$$S = p \cdot r = \sqrt{p(p-a)(p-b)(p-c)}$$

From this equality we obtain

$$r^2 = \frac{(p-a)(p-b)(p-c)}{p}$$

$$\frac{1}{r^2} = \frac{p}{(p-a)(p-b)(p-c)}$$

Using this identity, we transform the statement of the problem equivalently into

$$\frac{1}{(p-a)^2} + \frac{1}{(p-b)^2} + \frac{1}{(p-c)^2} \geqslant \frac{p}{(p-a)(p-b)(p-c)}$$

This inequality follows from the inequality for sequences of real numbers: $x^2 + y^2 + z^2 \geqslant xy + yz + zx$. Indeed, let $x = \frac{1}{p-a}$, $y = \frac{1}{p-b}$, $z = \frac{1}{p-c}$. We obtain

$$(\frac{1}{p-a})^2 + (\frac{1}{p-b})^2 + (\frac{1}{p-c})^2 \geqslant \frac{1}{(p-a)(p-b)} + \frac{1}{(p-b)(p-c)} + \frac{1}{(p-c)(p-a)}$$

$$\left(\frac{1}{p-a}\right)^2 + \left(\frac{1}{p-b}\right)^2 + \left(\frac{1}{p-c}\right)^2 \geqslant \frac{(p-c)+(p-b)+(p-a)}{(p-a)(p-b)(p-c)}$$

Since p is the semiperimeter, we have $p-c+p-b+p-a = 3p-(a+b+c) = 3p-2p = p$, and hence

$$\left(\frac{1}{p-a}\right)^2 + \left(\frac{1}{p-b}\right)^2 + \left(\frac{1}{p-c}\right)^2 \geqslant \frac{p}{(p-a)(p-b)(p-c)},$$

which proves the claim.

Method 2: We can also interpret the inequality

$$\frac{1}{(p-a)^2}+\frac{1}{(p-b)^2}+\frac{1}{(p-c)^2}\geqslant\frac{p}{(p-a)(p-b)(p-c)}$$

geometrically. Let $x, y, z > 0$ denote the tangent segments to the incircle of the triangle with sides a, b, c, drawn from the vertices between sides b and c, c and a, and a and b, respectively. From the tangent segment theorem, we know that $p = x+y+z$ and $p-a = x$, $p-b = y$, $p-c = z$. Thus, the transformed inequality becomes

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geqslant\frac{x+y+z}{xyz}$$

$$xyz\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\geqslant x+y+z$$

$$\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\geqslant x+y+z$$

$$2\left(\frac{yz}{2x}\right)+2\left(\frac{zx}{2y}\right)+2\left(\frac{xy}{2z}\right)\geqslant x+y+z$$

We estimate pairs of terms using the AM–GM inequality (since all values are positive).

$$\frac{yz}{2x}+\frac{zx}{2y}\geqslant 2\sqrt{\frac{xyz^2}{4xy}}=z$$

$$\frac{zx}{2y}+\frac{xy}{2z}\geqslant 2\sqrt{\frac{x^2yz}{4yz}}=x$$

$$\frac{xy}{2z}+\frac{yz}{2x}\geqslant 2\sqrt{\frac{xy^2z}{4zx}}=y$$

Adding these inequalities pairwise yields exactly the transformed statement, hence the inequality holds.