Problem 1. Solve in positive real numbers x, y, z the system of equations:

$$\begin{cases} xy + xy^2z = 2 \\ yz + xyz^2 = 2 \\ zx + x^2yz = 2 \end{cases}$$

Problem author: Michał Fronczek

Solution: Dividing the given equations respectively by the positive expressions $x + xyz$, $y + xyz$, and $z + xyz$, we obtain:

$$\begin{cases} y = \dfrac{2}{x + xyz} \\ z = \dfrac{2}{y + xyz} \\ x = \dfrac{2}{z + xyz} \end{cases}$$

Consider a solution (x, y, z) of the given system. Take the function $f(a) = \dfrac{2}{a + xyz}$. It is a harmonic function shifted to the left (since $xyz > 0$) by xyz. Thus, on the interval of positive real numbers, it is monotonic decreasing. Our system now has the form:

$$\begin{cases} y = f(x) \\ z = f(y) \\ x = f(z) \end{cases}$$

Assume first that $x > y$. Then, since f is decreasing, we have $f(x) < f(y)$, i.e. $y < z$. Then $f(y) > f(z)$, i.e. $z > x$, and finally $f(z) < f(x)$, i.e. $x < y$. Contradiction. An analogous issue appears if $x < y$. Therefore, $x = y$, and proceeding similarly for the other variables, we obtain $x = y = z$. Substituting this into one of the initial equations we get:

$$x^2 + x^4 = 2$$

$$(x^2 - 1)(x^2 + 2) = 0$$

Thus $x^2 = 1$ or $x^2 = -2$. The second option is an obvious contradiction (since a square is non-negative), so $x^2 = 1$ and since $x > 0$, we obtain $x = 1$. Hence, $x = y = z = 1$. It remains to notice that this option satisfies the conditions.

Answer: $x = y = z = 1$

Problem 2. Find all polynomials $P(x)$ with integer coefficients such that for every positive integer n, $P(n) > 0$ and $P(P(n) + n)$ is a prime number.

Problem author: Michał Fronczek
Solution: First observe that if P were constant, then it must be a prime number. Such an option obviously works. Assume therefore that P is not constant. First, note that from the problem assumptions we have $P(n) + n > 0$, since both summands are positive. Therefore $P(P(n) + n) > 0$.

There is a known fact that for a polynomial W with integer coefficients and for distinct integers a and b, the number $a - b$ divides $W(a) - W(b)$. It follows from the fact that $W(a) - W(b)$ is in fact a sum of expressions of the form $a_i(a^i - b^i)$ for integer a_i, and each $a^i - b^i$ is divisible by $a - b$. Since P has integer coefficients, $P(n)$ is an integer, hence so is $P(n)+n$, and therefore also $P(P(n)+n)$. Thus, for $a = P(n) + n$ and $b = n$, the expression $(P(n) + n) - n = P(n)$ divides $P(P(n) + n) - P(n)$. But $P(n)$ divides itself, so we get that $P(n)$ divides $P(P(n) + n)$. By assumption, however, the latter is a prime, so either $P(n) = 1$ or $P(P(n) + n) = P(n)$. Since P is not constant, from some point on (as n increases toward $+\infty$) it must become monotonic and tend to $+\infty$ or $-\infty$. But by assumption, $P(n)$ is positive, so the second option is excluded. Therefore, there exists an N such that for all $n > N$, P is monotonic, increasing to infinity, and hence greater than 1. Take such $n > N$. Then $P(n) > 1$, so we must have $P(P(n) + n) = P(n)$. However, $P(n) > 0$, so $P(n) + n > n > N$, and therefore by monotonicity $P(P(n) + n) > P(n)$, which contradicts the previously stated equality. Hence for $\deg(P) > 0$ we obtain a contradiction.

Answer: P is a constant polynomial equal to a prime number.