Problem 1. Determine whether there exist positive real numbers a, b, c satisfying the conditions

$$\begin{cases} abc \leqslant (\sqrt{2}-1)^3, \\ \frac{1}{a} + b < \sqrt{2} + 2, \\ \frac{1}{b} + c < \sqrt{2} + 1, \\ \frac{1}{c} + a < \sqrt{2}. \end{cases}$$

Author: Bartosz Trojanowski

Solution: Adding the second, third, and fourth inequalities, we obtain

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 3(\sqrt{2}+1) - a - b - c.$$

By the inequality between the geometric and harmonic means, we have

$$\sqrt{2} - 1 \geqslant \sqrt[3]{abc} \geqslant \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} > \frac{3}{3(\sqrt{2}+1) - a - b - c}.$$

Multiplying this inequality by $\sqrt{2}+1$, we obtain

$$3(\sqrt{2}+1) - a - b - c > 3(\sqrt{2}+1) \iff a + b + c < 0.$$

This contradicts the fact that a, b, c are positive. Therefore, such numbers do not exist. ■

Problem 2. Let $a, b, c > 0$ and $a + b + c = 3$. Prove that

$$\sqrt{\frac{a}{a+1}} + \sqrt{\frac{b}{b+1}} + \sqrt{\frac{c}{c+1}} \leqslant \frac{3\sqrt{3}}{2}.$$

Author: Tomasz Kossakowski
Solution: Let us consider the function

$$f(x) = \sqrt{\frac{x}{x+1}}, \quad x > 0.$$

We compute the second derivative:

$$f''(x) = \frac{-1 - 4x}{4\left(\frac{x}{1+x}\right)^{\frac{3}{4}}(1+x)^4} < 0 \quad \text{for } x > 0.$$

Therefore, the function f is concave on the interval $(0, \infty)$.

We apply Jensen's inequality for a concave function:

$$\frac{f(a) + f(b) + f(c)}{3} \leqslant f\left(\frac{a+b+c}{3}\right).$$

Since $a + b + c = 3$, we obtain

$$\frac{f(a) + f(b) + f(c)}{3} \leqslant f(1) = \sqrt{\frac{1}{2}}.$$

After multiplying by 3, we get

$$\sqrt{\frac{a}{a+1}} + \sqrt{\frac{b}{b+1}} + \sqrt{\frac{c}{c+1}} \leqslant 3 \cdot \sqrt{\frac{1}{2}} = \frac{3\sqrt{2}}{2}.$$

Equality holds if and only if $a = b = c = 1$.