Problem 1. Prove that there exists $A \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ we have

$$\left(1+\frac{1}{n}\right)^n \leqslant A.$$

Source: a well-known and well-liked problem
Selected by: Tomasz Kossakowski
Solution: Expanding the expression using the binomial theorem, we obtain:

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}.$$

For every $k \geqslant 0$ we have the estimate

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \leqslant \frac{n^k}{k!},$$

and hence

$$\binom{n}{k} \frac{1}{n^k} \leqslant \frac{1}{k!}.$$

Therefore,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \leqslant \sum_{k=0}^{n} \frac{1}{k!} \leqslant \sum_{k=0}^{\infty} \frac{1}{k!} = e.$$

Thus, we may take $A = e$, which completes the proof.

Problem 2. Let O be a point inside an acute triangle ABC. Let p, q be two distinct lines passing through O, both intersecting the sides AB and AC. Denote

$$K = p \cap AB, \quad L = p \cap AC, \quad M = q \cap AC, \quad N = q \cap AB.$$

The circumcircles of triangles NKO and MOL intersect again at a point $P \neq O$. Assume that the points A, N, K, B and A, L, M, C lie on the sides AB and AC respectively in this order. Prove that

$$\sphericalangle BAC = \sphericalangle PKL + \sphericalangle PMN.$$

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Solution: Observe that

$$\sphericalangle AMP = \sphericalangle PML = 180^\circ - \sphericalangle POL = \sphericalangle POK = \sphericalangle PNK = 180^\circ - \sphericalangle PNA.$$

Hence the quadrilateral $ANPM$ is cyclic. We also have

$$\sphericalangle PLK = \sphericalangle PLO = \sphericalangle PMO = \sphericalangle PMN = \sphericalangle PAN = \sphericalangle PAK,$$

so the quadrilateral $AKPL$ is also cyclic. Therefore,

$$\sphericalangle BAC = \sphericalangle BAP + \sphericalangle CAP = \sphericalangle NAP + \sphericalangle LAP = \sphericalangle PMN + \sphericalangle PKL,$$

which completes the proof. ■