Problem 1. Determine whether a knight can visit every square of a 5×5 chessboard exactly once and return to the starting square. What about a 4×4 chessboard?

Source: Wikipedia
Task selection: Maria Janyska
Solution: Let us begin with the 5×5 board. We will show that a knight cannot visit every square exactly once and return to the starting square, i.e. that there is no Hamiltonian cycle where the squares are treated as vertices of a graph and the knight's moves as edges.

Observe that with every move the knight passes from a black square to a white one or vice versa; it never lands twice in a row on a square of the same color.

The 5×5 chessboard has 25 squares, so in order for the knight to visit each square exactly once and return to the start, it must make exactly 25 moves. However, after an odd number of moves the knight always lands on a square of opposite color from the one it started on; thus after 25 moves it cannot be on a square of the same color, in particular not on the starting square. This is a contradiction.

Note that this observation tells us that for any $n \times m$ board with both m and n odd, there cannot exist a Hamiltonian cycle for a knight.

Thus on a 5×5 board no Hamiltonian cycle exists.

Now let us consider the 4×4 chessboard. In this part we again use the interpretation of the board as a graph described above. We want to show that a Hamiltonian cycle on such a board also does not exist.

Assume that a Hamiltonian cycle exists, so the chessboard can be represented by a graph containing such a cycle. We will use a lemma from graph theory whose proof appears at the end of this solution. By Lemma 1, for a graph containing a Hamiltonian cycle, after removing k vertices (together with incident edges), the graph breaks into at most k connected components. Thus we may remove the 4 central squares of our chessboard and count the number of remaining connected components. There are 6 of them—after linking each vertex to all squares it can move to, we obtain four isolated vertices and two cycles of four vertices each located on the boundary. Since $4 < 6$, we obtain a contradiction, because if a Hamiltonian cycle existed, there would be at most 4 connected components.

Thus we also obtain a contradiction for the 4×4 chessboard.

Lemma 1. For a graph containing a Hamiltonian cycle, after removing k vertices (together with their incident edges), the graph can be divided into at most k connected components.

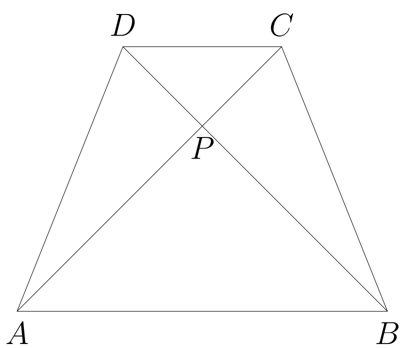
Assume to the contrary that after removing k vertices, the graph breaks into l connected components with $l > k$. We start adding back the removed vertices along with their edges. By the definition of a Hamiltonian cycle, each vertex is visited exactly once, so we may pass from one component to another only through the added vertices, but each such vertex can be used only once. We need to connect the l components into a closed cycle, i.e. fill l "gaps" between them. When we add one vertex, we can connect at most two components, so after adding k vertices there remain $l - k \geqslant 1$ gaps, which means it is not a cycle. This contradiction proves the bound $l \leqslant k$.

Problem 2. In a trapezoid $ABCD$ with bases AB, CD, the diagonals intersect at point P. Given that $[CPD] = 8, [APD] = 16$, compute the area of the entire trapezoid.

Remark: The notation $[ABC]$, i.e. the name of a polygon in square brackets, denotes the area of that polygon.

Source: School Stage of the Regional Mathematics Contest organized by the Łódź School Superintendent for primary school students, 2025/26.
Task selection: Maria Janyska

Solution:



Solution 1. Since the bases AB, CD are parallel, we may write the equality of areas $[ADB] = [ACB]$. After subtracting the common part, i.e. $[APB]$, we get $[BPC] = [APD] = 16$.

Triangles CPD and CAD share the same base and have heights parallel to each other. Since we know their areas, we can conclude that the ratio of heights onto side CD is $1 : 3$, which by parallelism gives that the height of triangle APB from vertex P equals the difference between the height of CAD onto CD and the height of CPD from P. Thus the ratio of the considered heights CPD to APB is $1 : 2$.

Triangles APB and CPD are similar, since angles $\sphericalangle ABP$ and $\sphericalangle CDP$ are alternate interior angles, and likewise $\sphericalangle BAP$ and $\sphericalangle DCP$. Thus the ratio of their areas equals the ratio of the squares of their corresponding heights. Since this ratio is $2 : 1$, the ratio of areas is $4 : 1$, and because $[CPD] = 8$, we get $[ABP] = 32$.

Hence the total area of the trapezoid is

$$[ABCD] = [APB] + [APD] + [CPD] + [BPC] = 8 + 16 + 32 + 16 = 72.$$

Solution 2. As in the previous solution, we first show that $[BPC] = 16$.

Observe that triangles APD and CPD share the same height from point D. Knowing their areas, we conclude that the ratio of their bases is $AP : PC = 2 : 1$.

Similarly, triangles APB and CPB share the same height from point B. We already know the ratio of their bases and the area of one of them, equal to 16. The ratio of their areas equals the ratio of the bases onto which the common height is dropped, so:

$$[APB] : [CPB] = AP : CP = 2 : 1.$$

Therefore $[APB] = 32$.

Adding the four areas yields the same result:

$$[ABCD] = 72.$$