Problem 1. Problem 1. Solve the system of equations:

$$\begin{cases} x + y = xy, \\ x^2 + y^2 = 10. \end{cases}$$

Author: Tomasz Kossakowski
Solution: Let

$$s = x + y, \qquad p = xy.$$

From the first equation we have $s = p$. Using the identity $x^2 + y^2 = s^2 - 2p$, we obtain

$$s^2 - 2s = 10,$$

so

$$s^2 - 2s - 10 = 0.$$

Solving this quadratic equation:

$$s = \frac{2 \pm \sqrt{4 + 40}}{2} = 1 \pm \sqrt{11}.$$

Since $p = s$, we have two cases:

$$(s, p) = (1 + \sqrt{11},\, 1 + \sqrt{11}) \quad \text{or} \quad (1 - \sqrt{11},\, 1 - \sqrt{11}).$$

The numbers x and y are roots of the quadratic equation (by Vieta's formulas)

$$t^2 - st + p = 0,$$

that is,

$$t^2 - st + s = 0.$$

Hence

$$x, y = \frac{s \pm \sqrt{s^2 - 4s}}{2} = \frac{s \pm \sqrt{s(s - 4)}}{2}.$$

Case 1. $s = 1 + \sqrt{11}$

$$s(s - 4) = (1 + \sqrt{11})(-3 + \sqrt{11}) = 8 - 2\sqrt{11} = 2(4 - \sqrt{11}),$$

thus

$$x, y = \frac{1 + \sqrt{11} \pm \sqrt{2(4 - \sqrt{11})}}{2}.$$

Case 2. $s = 1 - \sqrt{11}$

$$s(s-4) = (1-\sqrt{11})(-3-\sqrt{11}) = 8 + 2\sqrt{11} = 2(4+\sqrt{11}),$$

so

$$x, y = \frac{1-\sqrt{11} \pm \sqrt{2(4+\sqrt{11})}}{2}.$$

Both cases yield real solutions, since the expressions under the square root are positive.

Answer:

$$\boxed{\begin{aligned} &(x,y) = \left(\frac{1+\sqrt{11} \pm \sqrt{2(4-\sqrt{11})}}{2}, \frac{1+\sqrt{11} \mp \sqrt{2(4-\sqrt{11})}}{2}\right), \\ \text{or} \quad &(x,y) = \left(\frac{1-\sqrt{11} \pm \sqrt{2(4+\sqrt{11})}}{2}, \frac{1-\sqrt{11} \mp \sqrt{2(4+\sqrt{11})}}{2}\right). \end{aligned}}$$

Each pair may be interchanged, since the system is symmetric with respect to x and y. $\square$

Problem 2. Prove that there exist infinitely many positive integers n such that

$$\varphi(n) = \frac{n}{2}.$$

Note: By $\varphi(n)$ we denote the number of integers $0 < k < n$ such that $\gcd(k, n) = 1$.

Author: Bartosz Trojanowski
Solution: We will prove that the conditions of the problem are satisfied by every power of two. Indeed, then $\gcd(k, n) = 1$ if and only if k is odd, and among the numbers $1, 2, \ldots, n$ there are exactly $\frac{n}{2}$ such numbers.
Remark: It turns out that powers of two are the only numbers satisfying this property.