Problem 1. For every positive integer t, let $d(t)$ denote the smallest positive integer whose factorial is divisible by t (for example, $d(6) = 3$, since $3! = 2 \cdot 3$ is a multiple of 6, while $2!$ and $1!$ are not). Solve the equation

$$d(n) = \frac{n}{2}.$$

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Solution: Let $v_p(n)$ denote the p-adic valuation of n.
Clearly, n must be even. First, assume that n is a power of 2. Thus $d(2^k) = 2^{k-1}$ for some positive integer k. A direct check shows that $k = 1, 2, 4$ do not satisfy the condition, while $k = 3$ does. For $k \geqslant 5$, by Legendre's formula we have

$$v_2(2^{k-2}!) = \frac{2^{k-2}}{2} + \frac{2^{k-2}}{4} + \cdots \geqslant 2^k,$$

so $d(2^k) \leqslant 2^{k-2} < 2^{k-1}$. Hence, when n is a power of 2, the only solution is $n = 8$.
Now let $n = 2p$ for some odd prime p. Then it is clear that $d(n) = \frac{n}{2}$.
Finally, suppose n has at least two distinct odd prime divisors greater than 1. Let q be the second largest of these divisors. Then $q < \frac{n}{2}$, hence $d(n) \leqslant q < \frac{n}{2}$.
In conclusion, the equation is satisfied by all even numbers that are twice a prime, and by $n = 8$. ■

Problem 2. Let ABC be an acute-angled triangle and let the circle circumscribed about it be given. The tangents to the circle at points B and C meet at D, those at C and A meet at E, and those at A and B meet at F. Prove that the lines AD, BE, and CF are concurrent.

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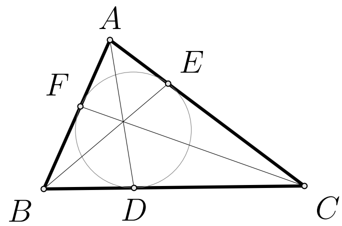
Solution:



Observe that the given circle is a conic, and that the lines AF, FB, BD, DC, CE, and EA are all tangent to it. Hence, the hexagon $AFBDCE$ is circumscribed about a conic, and by **Brianchon's theorem**, the lines AD, BE, and CF are concurrent.