Problem 1. Klara's math grades are as follows:

6 with weight 1

6 with weight 1

6 with weight 1

5 with weight 2

4 with weight 2

Today, the class is taking a major test with weight 3. The grading scale is standard: from 1 to 6. What grade does she need to get so that the weighted average of her grades is greater than or equal to 5.5?

Author: Scarlett Lafa
Answer: she cannot have such an average.
Solution: Klara's weighted average after the test will be as follows (let a denote her grade on the big test):

$$\frac{6\cdot 1+6\cdot 1+6\cdot 1+5\cdot 2+4\cdot 2+a\cdot 3}{1+1+1+2+2+3}=\frac{36+3a}{10}.$$

This average must be greater than or equal to 5.5, that is,

$$\frac{36+3a}{10}\geqslant 5.5 \quad /\cdot 10$$

$$36+3a\geqslant 55$$

$$3a\geqslant 19$$

$$a\geqslant 6.\overline{3}.$$

This is impossible, because the highest grade Klara can receive is 6.

Problem 2. For $a,\ b \geqslant 0$, prove that the following inequality holds:

$$2a^2 + 2b^2 + a + b \geqslant 2ab + 2\sqrt{ab}$$

Author: Scarlett Lafa
Solution: Note that for $a, b \geqslant 0$, $a^2 + b^2 \geqslant 2ab \geqslant ab$, hence

$$a^2 + b^2 \geqslant ab \quad / \cdot 2$$

$$2a^2 + 2b^2 \geqslant 2ab.$$

Also note that $(\sqrt{a} - \sqrt{b})^2 \geqslant 0$, therefore

$$a + b - 2\sqrt{ab} \geqslant 0$$

$$a + b \geqslant 2\sqrt{ab}.$$

Adding the two inequalities we obtain

$$2a^2 + 2b^2 + a + b \geqslant 2ab + 2\sqrt{ab}.$$