

 **Problem 1.** During a very boring lesson at school, Marek counted the hairs of three classmates sitting in front of him. He realized that the number of hairs was exactly the same as the number of words (meaningful or not) that can be formed by rearranging the letters of the word OWŁOSIENIE. How many hairs does each of Marek's classmates have on average?

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Solution: If the word *OWŁOSIENIE* had all distinct letters, then one could form $10!$ words from it. (The first position can be filled by 10 letters, the second by 9, etc.). But our word has three letters, each of which repeats twice. Therefore, in order to obtain the total number of hairs, we must divide $10!$ by $2! \cdot 2! \cdot 2!$. Why? Because for each of the letters O, I, E, every word has been counted $2!$ times. (That is, the letter O can occupy two positions in the word, so we counted every word $2!$ times).

Additional example: For the word *ALASKA*, if all the letters were different, we would have $6!$ possible words, but A is repeated 3 times, so we must divide $6!$ by $3!$, because A can be placed in three positions in the word, and we can “distribute” our three A's among these positions in $3!$ ways, so each word from *ALASKA* was counted $3!$ times.

Returning to the problem:

$$\frac{10!}{2! \cdot 2! \cdot 2!} = 453600.$$

We obtained the total number of hairs; now we just need to calculate the arithmetic mean.

$$\frac{453600}{3} = 151200$$

Answer: On average, each of Marek's classmates has 151200 hairs.

Problem 2. Let a, b be relatively prime positive integers. We call a pair of positive integers $c \neq a, d \neq b$ *humanistic*, if the following holds

$$a^2 + b^2 = ac + bd.$$

Prove that for every humanistic pair the inequality $a + b < c + d$ holds.

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Solution: Let c, d be a humanistic pair. If the inequalities $c > a, d > b$ hold, then $a + b < c + d$, so let us assume that at least one of these inequalities is false. Without loss of generality, assume that $a \geq c$. Then $c = a - k$ for some integer $k \geq 1$. Hence we have

$$a^2 + b^2 = ac + bd \implies b^2 + ak = bd.$$

Since the numbers a and b are relatively prime, we get $b \mid k$, which gives $k \geq b$, hence $c = a - k \leq a - b$. From this we obtain

$$a^2 + b^2 = ac + bd \leq a(a - b) + bd = a^2 - ab + bd \implies b^2 \leq -ab + bd \implies a + b \leq d.$$

Therefore $c + d > d \geq a + b$, which completes the proof. ■