Problem 1. On the board, the numbers

$$1, 2, 3, \ldots, 2025$$

are written. Scarlett and Antoni play a game. A move consists of replacing three numbers according to the rule

$$(a, b, c) \longrightarrow (a+b-c,\ b+c-a,\ c+a-b).$$

The winner is the player after whose move all the numbers on the board are equal. Scarlett moves first. Determine who has a winning strategy.

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Solution: We will show that this game cannot be won.
Assume for contradiction that it can. Notice that the sum of all numbers at any moment of the game is constant, since from numbers with sum $a+b+c$ we create numbers with sum

$$(a+b-c)+(b+c-a)+(c+a-b) = a+b+c.$$

Hence, the sum of all numbers is

$$\sum_{n=1}^{2025} n = \frac{1}{2} \cdot 2025 \cdot (2025+1) = 2025 \cdot 1013,$$

so we would want all numbers on the board to equal 1013 at the end. On the other hand, the sum of the squares of the numbers on the board does not decrease, which we show using the following inequality. From the squares $a^2+b^2+c^2$ we form

$$(a+b-c)^2+(b+c-a)^2+(c+a-b)^2,$$

so

$$(a+b-c)^2+(b+c-a)^2+(c+a-b)^2-(a^2+b^2+c^2) \geqslant 0.$$

Using the formula $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz$, we get:

$$(a^2+b^2+c^2+2ab-2bc-2ac)+(a^2+b^2+c^2-2ab+2bc-2ac) \\ +(a^2+b^2+c^2-2ab-2bc+2ac)-(a^2+b^2+c^2) \geqslant 0$$

Simplifying:

$$2a^2-2ab+2b^2-2ac-2bc+2c^2 = (a-b)^2+(b-c)^2+(c-a)^2 \geqslant 0.$$

If the sum of squares is non-decreasing, the minimal possible sum is

$$\sum_{n=1}^{2025} n^2 = \frac{1}{6} \cdot 2025 \cdot 2026 \cdot 5051 = 3453747525,$$

while the desired sum of squares would be

$$2025 \cdot 1013^2 = 2077992225,$$

so

$$\sum_{n=1}^{2025} n^2 > 2025 \cdot 1013^2,$$

which is a contradiction, because $\sum_{n=1}^{2025} n^2$ was supposed to be the minimal sum of squares. ■

Problem 2. Prove that for any positive real numbers x, y, z such that $(x+y)(y+z)(z+x) \neq 0$, the following inequality holds:

$$\frac{(1+x^2)(1+y^2)}{2(x+y)} + \frac{(1+y^2)(1+z^2)}{2(y+z)} + \frac{(1+z^2)(1+x^2)}{2(z+x)} \geqslant x+y+z.$$

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Solution: Notice that for any real numbers a, b we have

$$(1+a^2)(1+b^2) \geqslant (a+b)^2.$$

Indeed, expanding all terms gives the inequality

$$1 + a^2b^2 \geqslant 2ab,$$

which holds for all a, b. Hence,

$$\frac{(1+x^2)(1+y^2)}{2(x+y)} \geqslant \frac{(x+y)^2}{2(x+y)} = \frac{x+y}{2}.$$

Similarly, we bound the other two fractions. Adding all obtained inequalities side by side gives the desired result. ■