Problem 1. In trapezoid $ABCD$, points M and N are the midpoints of the legs BC and DA, respectively. Compute the ratio

$$\frac{P_{ABM}+P_{CDN}}{P_{ABCD}}.$$

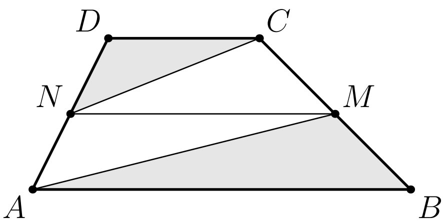
Author of the problem: Maria Janyska

Solution: The area of the entire trapezoid is clearly $\frac{1}{2}(a+b)\cdot 2h$. Now let us compute the areas of the desired triangles. Notice that the line MN is the midline of the trapezoid. This means it also divides the height of the trapezoid into two equal segments. These segments are perpendicular to the bases AB and CD, and thus are also parallel to the altitudes of triangles $\triangle ABM$ and $\triangle CDN$ drawn from the vertices M and N. Since $MN \parallel AB \parallel CD$, these altitudes have equal lengths of $\frac{2h}{2}=h$. Therefore, we can compute the sum:

$$P_{ABM}+P_{CDN}=\frac{1}{2}ah+\frac{1}{2}bh=\frac{1}{2}h(a+b).$$

Thus, we obtain the final result:

$$\frac{P_{ABM}+P_{CDN}}{P_{ABCD}}=\frac{\frac{1}{2}h(a+b)}{\frac{1}{2}(a+b)2h}=\frac{1}{2}.$$



Problem 2. Given that $a+b+c=0$ and $a,b,c \in \mathbb{Z}$, prove that

$$32(a^4+b^4+c^4)$$

is a perfect square.

Author of the problem: *Matematyka Olimpijska – Algebra i Teoria Liczb* by A. Neugebauer, Omega Publishing, Kraków 2022
Choice of problem: Maria Jansyka
Solution: Let us denote

$$S = 32(a^4+b^4+c^4).$$

Using Girard–Newton formulas, we can express S in terms of elementary symmetric polynomials of three variables (for more details, see the book *Matematyka Olimpijska – Algebra i Teoria Liczb* by A. Neugebauer, Omega Publishing – Chapter 6), as follows:

$$\begin{aligned} S = 32\big(((a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc)\cdot \\ \cdot(a+b+c)-(a^2+b^2+c^2)(ab+bc+ca)+(a+b+c)abc\big). \end{aligned}$$

Substitute the given condition $a+b+c=0$:

$$\begin{aligned} S = 32\left(-(a^2+b^2+c^2)(ab+bc+ca)\right) = \\ = -32\left(((a+b+c)^2-2ab-2bc-2ca)(ab+bc+ca)\right) = \\ = -32\cdot(-2)(ab+bc+ca)^2 = (8(ab+bc+ca))^2. \end{aligned}$$

Since $a,b,c \in \mathbb{Z}$, it follows that $8(ab+bc+ca) \in \mathbb{Z}$, and therefore S is a perfect square of an integer, as required.