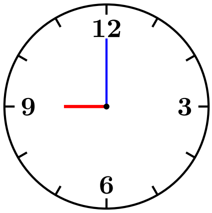
Solutions for problems from 06/19/2025

You can expect more tasks as soon as tomorrow!

Problem 1. Imagine using a traditional clock and the time is 9:00. After how much time will the clock hands form a 90° angle?



Problem author: Tomasz Kossakowski

Solution: After $32\frac{8}{11}$ minutes.

In one hour, the minute hand (let's say it's blue) moves 360°, so in one minute it moves 6°.

The hour hand (let's say it's red) moves $\frac{360}{12} = 30^\circ$ per hour, i.e., 0.5° per minute.

At 9:00, the blue hand is at 0°, and the red hand is at $(0^\circ - 90^\circ)$. After t minutes, their positions are $6t^\circ$ and $0.5t^\circ - 90^\circ$. So we are solving:

$$|\,6t - (0.5t - 90)\,| = 270.$$

Which gives $5.5t + 90 = 270$ or $5.5t + 90 = -270$, i.e. $5.5t = 180$, hence $t = 32\frac{8}{11}$, since the second equation gives a negative time. ■

Problem 2. Solve the equation in real numbers:

$$x^{2025} + x^{2024}y + x^{2023}y^2 + \ldots + x^2y^{2023} + xy^{2024} + y^{2025} = 0.$$

Problem author: Tomasz Kossakowski
Solution: From the formula for the difference of powers

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$$

we can infer that if $x - y \neq 0$ (we'll check this case separately), then:

$$x^{2025} + x^{2024}y + \ldots + xy^{2024} + y^{2025} = \frac{x^{2026} - y^{2026}}{x - y} = 0,$$

so $x^{2026} = y^{2026}$, which means $|x| = |y|$, so either $x = y$ or $x = -y$.
If $x = y$, the equation becomes:

$$2026x^{2025} = 0 \implies (x, y) = (0, 0).$$

If $x = -y$, the equation becomes:

$$x^{2025} - x^{2025} + x^{2025} - \ldots - x^{2025} + x^{2025} - x^{2025} = 0 \iff 0 = 0,$$

so all (x, y) such that $x = -y$ are solutions. ■

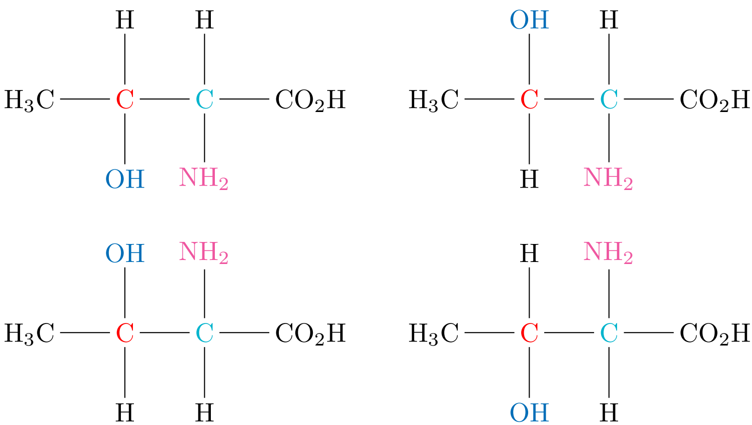
Problem 3. ***Stereoisomers*** are molecules with the same chemical composition (same types and counts of atoms), and the same connections between atoms, but differ in the 3D arrangement of those atoms.

Imagine it like this: you have two LEGO figures made of exactly the same bricks. Each brick is identical and in the same quantity. But one figure has its arm extended to the right, and the other to the left. Same pieces, different look – and possibly different behaviors, such as how they fit with other parts.

In chemistry, stereoisomers can also behave differently – though they have the same "recipe" (chemical formula), their properties may vary due to the spatial configuration: taste, smell, or even effects in the body.

In organic chemistry, each carbon has 4 bonds. Below is a chemical compound called 2-amino-3-hydroxybutanoic acid (the name is not important). Note that it has 4 stereoisomers (assume mirror images count as different). A "group" is any string of atoms connected by bonds, which we treat as a single entity here.

At the red carbon C, the OH group can be on top or bottom. At the turquoise carbon C, the NH_2 group can also be on top or bottom.

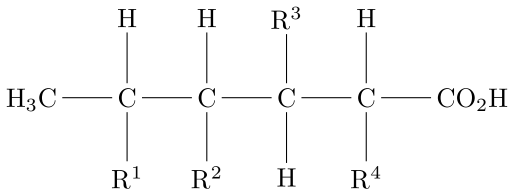


How many stereoisomers does a hypothetical compound have which consists of n such carbon atoms between the groups:

$$H_3C \text{ —— } \dots \text{ —— } CO_2H$$

Where from each carbon i there is exactly one hydrogen H and one group R^i, and all R^1, R^2, ..., R^n are pairwise distinct and different from H, H_3C and CO_2H?

To visualize this, below is the structure for $n = 4$:



Problem author: Tomasz Kossakowski

Solution: For each R^i group on the i-th carbon atom, we have two placement options – top or bottom. Using the multiplication rule, the number of stereoisomers is:

$$\underbrace{2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}_{n \text{ times}} = 2^n.$$

■

Problem 4. For which values of $n \in \mathbb{Z}_+$ does the expression

$$\frac{4n^4 - 48n^3 + 168n^2 - 144n + 60}{n^2 - 6n + 3}$$

yield a positive integer?

Problem author: Tomasz Kossakowski
Solution:
Method 1: Let us divide the polynomial $4n^4 - 48n^3 + 168n^2 - 144n + 60$ by the polynomial $n^2 - 6n + 3$:

$$\begin{array}{r|rrrrr}
 & 4n^2 & -\ 24n & +\ 12 & & \\
\hline
n^2 - 6n + 3 & 4n^4 & -\ 48n^3 & +\ 168n^2 & -\ 144n & +\ 60 \\
 & 4n^4 & -\ 24n^3 & +\ 12n^2 & & \\
\hline
 & & -24n^3 & +\ 156n^2 & -\ 144n & \\
 & & -24n^3 & +\ 144n^2 & -\ 72n & \\
\hline
 & & & 12n^2 & -\ 72n & +\ 60 \\
 & & & 12n^2 & -\ 72n & +\ 36 \\
\hline
 & & & & & 24
\end{array}$$

From this, it follows that:

$$\frac{4n^4 - 48n^3 + 168n^2 - 144n + 60}{n^2 - 6n + 3} = 4n^2 - 24n + 12 + \frac{24}{n^2 - 6n + 3}$$

Notice that for this expression to be an integer, $n^2 - 6n + 3$ must be a divisor of 24. For $n = \{1, 2, 3, 4, 5\}$, the expression $n^2 - 6n + 3$ takes negative values, while for larger natural numbers it becomes positive.
The divisors of 24 to consider are:

$$D_{24} = \{\pm 1, \pm 2, \pm 3, \pm 4, 6, 8, 12, 24\}$$

We also know that $\frac{4n^4 - 48n^3 + 168n^2 - 144n + 60}{n^2 - 6n + 3}$ is positive for integers greater than 5. Let's now look for valid values of n. Instead of checking all divisors, let's search for those that are potential candidates.
Consider the equation:

$$n^2 - 6n + 3 = k \iff n^2 - 6n + (3 - k) = 0,$$

where $k \in D_{24}$. We want to find those integer values of k for which the equation has integer solutions n. In that case, the discriminant must be a perfect square:

$$\Delta = (-6)^2 - 4 \cdot 1 \cdot (3-k) = 36 - 4(3-k) = 36 - 12 + 4k = 24 + 4k$$

$$k = \frac{m^2 - 24}{4} \quad \text{for some even } m \in \mathbb{Z}_+$$

Only $k = -2$ and $k = 3$ satisfy this condition. Then:

$$n^2 - 6n + 5 = 0 \iff n = 1 \text{ or } n = 5$$

$$n^2 - 6n = 0 \iff n = 0 \text{ or } n = 6$$

Among these, the only suitable value is $n = 6$, which is the solution to our problem. ■

Method 2:
We want to decompose the numerator as:

$$(n^2 - 6n + 3)(An^2 + Bn + C) + D$$

Now expand the product:

$$(n^2-6n+3)(An^2+Bn+C) = An^4+(B-6A)n^3+(C-6B+3A)n^2+(-6C+3B)n+3C$$

Adding the remainder D, we get:

$$An^4 + (B-6A)n^3 + (C-6B+3A)n^2 + (-6C+3B)n + (3C+D)$$

We compare the coefficients with those in the numerator:

$$\begin{aligned} A &= 4 \\ B - 6A = -48 &\Rightarrow B = -24 \\ C - 6B + 3A = 168 &\Rightarrow C = 12 \\ -6C + 3B &= -144 \quad \text{(satisfied)} \\ 3C + D = 60 &\Rightarrow D = 24 \end{aligned}$$

Substituting everything in, we obtain:

$$\frac{4n^4 - 48n^3 + 168n^2 - 144n + 60}{n^2 - 6n + 3} = 4n^2 - 24n + 12 + \frac{24}{n^2 - 6n + 3}$$

Note that for this expression to be an integer, $n^2 - 6n + 3$ must divide 24. For $n = \{1, 2, 3, 4, 5\}$ the expression $n^2 - 6n + 3$ takes negative values; for larger natural numbers, it is positive.
The divisors of 24 to consider are:

$$D_{24} = \{\pm 1, \pm 2, \pm 3, \pm 4, 6, 8, 12, 24\}$$

Additionally, we know that the full expression is positive for integers greater than 5. Let's now search for valid values of n. Instead of testing all divisors, we will look for those k such that:

$$n^2 - 6n + 3 = k \iff n^2 - 6n + (3 - k) = 0$$

where $k \in D_{24}$. We want to find such integer values of k for which this quadratic equation has integer solutions n. Therefore, the discriminant must be a perfect square:

$$\Delta = (-6)^2 - 4 \cdot 1 \cdot (3 - k) = 36 - 4(3 - k) = 36 - 12 + 4k = 24 + 4k$$

$$k = \frac{m^2 - 24}{4} \quad \text{for some even } m \in \mathbb{Z}_+$$

Only $k = -2$ and $k = 3$ satisfy this condition. Then:

$$n^2 - 6n + 5 = 0 \iff n = 1 \text{ or } n = 5$$

$$n^2 - 6n = 0 \iff n = 0 \text{ or } n = 6$$

From these, the only valid value (positive and greater than 5) is $n = 6$, which is the solution to our problem. ■